Pergamon

Appl. Math. Lett. Vol. 11, No. 2, pp. 81–84, 1998

0893-9659/98 $19.00 + 0.00

PII: S0893-9659(98)00015-9

Global Stability for a Lotka-Volterra System with a Weakly Diagonally Dominant Matrix

ZHENGYI LU
Department of Mathematics, Sichuan University
Chengdu 610064, P.R. China

(Received May 1997; accepted June 1997)

Communicated by S. Levin

Abstract—The unique positive equilibrium of a Lotka-Volterra system with a weakly diagonally dominant interaction matrix is globally stable.

Keywords—Lotka-Volterra system, LaSalle's invariance principle, Global stability.

We wish to study here the global stability of the Lotka-Volterra system,

$$\dot{x}_i(t) = x_i(t)\left[r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right], \qquad i = 1, \dots, n. \tag{1}$$

The x_i denote the densities; the r_i are intrinsic growth rates, and the a_{ij} describe the effect of the j^{th} upon the i^{th} population. The matrix $A = (a_{ij})_{n\times n}$ is called the interaction matrix.

System (1) is assumed to have a unique positive equilibrium $x^* = (x_1^*, \dots, x_n^*)$, which in turn implies that

$$\det(A) \neq 0. \tag{2}$$

The global stability of system (1) has been studied by many authors [1–3]. It is shown that the diagonal dominance of the interaction matrix A implies the global stability of the system.

DEFINITION.

(i) *Matrix $A = (a_{ij})_{n\times n}$ is said to satisfy the diagonally dominant condition (DD), if there exist positive constants α_i ($i = 1, \dots, n$) such that*

$$(\mathrm{DD})\colon \alpha_i a_{ii} + \sum_{j=1, j\neq i}^{n} \alpha_j |a_{ji}| < 0, \qquad i = 1, \dots, n. \tag{3}$$

Project supported by the Foundation of the State Education Commission of China and the National Natural Science Foundation of China.
The author is grateful to the referees for their suggestions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

(ii) *A satisfies the weakly diagonally dominant condition (WDD), if there exist positive constants α_i $(i = 1, \dots, n)$ such that*

$$\text{(WDD)}: \alpha_i a_{ii} + \sum_{j=1, j\neq i}^{n} \alpha_j |a_{ji}| \leq 0, \qquad i = 1, \dots, n. \tag{4}$$

Note that when all the equalities in (4) hold, we have

$$\det\left(\bar{A}\right) = 0, \tag{5}$$

where $\bar{A} = (\bar{a}_{ij})$ with $\bar{a}_{ii} = a_{ii}$ and $\bar{a}_{ij} = |a_{ij}|$ for $i, j = 1, \dots, n;\ i \neq j$.

The aim of the present paper is to show that condition (WDD) instead of condition (DD) is enough to ensure the global stability of system (1).

LEMMA. *If A is irreducible (i.e., the linear transformation A does not map into itself any nonzero proper linear subspace spanned by a subset of the standard basis vectors) and satisfies (WDD), then either A satisfies (DD) or all the equalities in (4) hold true.*

PROOF. Suppose one of the inequalities is strict, say, the first one, then change α_1 to be $\bar{\alpha}_1 = \alpha_1 - \delta_1$ ($\delta_1 > 0$ small). Since A is irreducible, there is an inequality, besides the first one, which is strict, say, the second one. Change α_2 to be $\bar{\alpha}_2 = \alpha_2 - \delta_2$ ($\delta_2 > 0$ small). By the irreducibility of A, we can obtain positive constants $\bar{\alpha}_3, \dots, \bar{\alpha}_n$, so that (3) holds true provided α_i in (4) are changed to be $\bar{\alpha}_i$.

This completes the proof of the lemma.

The main result of this paper is as follows.

THEOREM. *If A satisfies condition (WDD), then the positive equilibrium x^* of system (1) is globally stable.*

PROOF. Define a Liapunov function

$$\dot{V}(t) = \sum_{i=1}^{n} \alpha_i \left| \ln \frac{x_i}{x_i^*} \right|,$$

where α_i, $i = 1, \dots, n$, satisfy (4).

Then the *Dini* derivative of $V(t)$ along a solution $x(t)$ of (1) takes the form

$$\begin{aligned} DV(t) &= \sum_{i=1}^{n} \alpha_i \epsilon_i \frac{\dot{x}_i}{x_i} = \sum_{i=1}^{n} \alpha_i \epsilon_i \sum_{j=1}^{n} a_{ij} \left(x_j - x_j^*\right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1, i\neq j}^{n} \alpha_i \epsilon_i \epsilon_j a_{ij} \left|x_j - x_j^*\right| \right) + \sum_{j=1}^{n} \alpha_j a_{jj} \left|x_j - x_j^*\right| \\ &= \sum_{j=1}^{n} \left(\alpha_j a_{jj} + \sum_{i=1, i\neq j}^{n} \alpha_i |a_{ij}| \right) \left|x_j - x_j^*\right| + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} \alpha_i (\epsilon_i \epsilon_j a_{ij} - |a_{ij}|) \left|x_j - x_j^*\right|, \end{aligned} \tag{6}$$

where

$$\epsilon_i = \begin{cases} 1, & \text{for } x_i(t) \geq x_i^*, \\ -1, & \text{for } x_i(t) < x_i^*, \end{cases}$$

when $DV(t)$ denotes the upper-right derivative, or

$$\epsilon_i = \begin{cases} -1, & \text{for } x_i(t) > x_i^*, \\ 1, & \text{for } x_i(t) \leq x_i^*, \end{cases}$$

when $DV(t)$ denotes the upper-left derivative.

Now suppose A is irreducible.

If A is diagonally dominant, then $\dot{V} < 0$ for all $x \neq x^*$. Hence, x^* is globally stable in this case.

Consider the case that all the equalities in (4) hold. By the LaSalle's invariance principle [4], the LaSalle's invariant set M is contained in E:

$$E = \left\{ x(t) \in R_+^n \;\middle|\; \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} \alpha_i(\epsilon_i\epsilon_j a_{ij} - |a_{ij}|) \left|x_j - x_j^*\right| = 0 \right\}$$
$$= \left\{ x(t) \in R_+^n \;\middle|\; (\epsilon_i\epsilon_j a_{ij} - |a_{ij}|) \left|x_j - x_j^*\right| = 0,\ i,j = 1,\dots,n;\ i \neq j \right\}.$$

Now we show, by induction on the species number k $(1 \leq k \leq n)$, that M is identical with the unique positive equilibrium x^*.

Suppose $x(t) \in M \subset E$.

If $x_i(t) - x_i^* \neq 0$ for each $i \in \{1,\dots,n\}$, then $\epsilon_i = 1$ or -1 for all $t \geq 0$. By the structure of E, it follows that $\epsilon_i\epsilon_j a_{ij} = |a_{ij}|$ for $i,j = 1,\dots,n; i \neq j$. In this case,

$$\begin{aligned} \det\left(\bar{A}\right) &= \det\left(\bar{a}_{ij}\right) = \det\left(\epsilon_i\epsilon_j a_{ij}\right) \\ &= \sum (-1)^{\tau(i_1 i_2 \cdots n)} a_{1i_1} \cdot a_{2i_2} \cdots a_{ni_n} \cdot \epsilon_1 \cdot \epsilon_{i_1} \cdot \epsilon_2 \cdot \epsilon_{i_2} \cdots \epsilon_n \cdot \epsilon_{i_n} \\ &= \sum (-1)^{\tau(i_1 i_2 \cdots n)} a_{1i_1} \cdot a_{2i_2} \cdots a_{ni_n} = \det\left(a_{ij}\right) = \det\left(A\right). \end{aligned} \tag{7}$$

Clearly, (7) together with (2) and (5) leads to a contradiction. This shows the case of $k = 1$.

We suppose, without loss of generality, that in the LaSalle's invariant set M, we have that $x_i(t) - x_i^* = 0$ for each $i \in I = \{1,\dots,k-1\}$, but $x_j(t) - x_j^* \neq 0$ for each $j \in J = \{k,\dots,n\}$. Since $x(t) \in E$ and the interaction matrix A is irreducible, we have, for upper-right derivative (6), that

$$(a_{ij} - |a_{ij}|) \left|x_j\left(\bar{t}\right) - x_j^*\right| = 0,$$

or, for upper-left derivative (6), that

$$(-a_{ij} - |a_{ij}|) \left|x_j\left(\bar{t}\right) - x_j^*\right| = 0.$$

By the irreducibility of A, there is one $i \in I$ and one $j \in J$ such that $a_{ij} \neq 0$, say, $a_{1k} \neq 0$. Clearly, we can always choose $DV(t)$ in (6) as upper-right derivative or upper-left one such that one of $(a_{1k} - |a_{1k}|)$ and $(-a_{1k} - |a_{1k}|)$ is nonzero. This implies that $x_k(\bar{t}) - x_k^* = 0$ in M. By the induction assumption, we have

$$x_i\left(\bar{t}\right) - x_i^* = 0, \qquad \text{for } i = 1,\dots,n. \tag{8}$$

By the uniqueness of the solutions, we obtain

$$x_i(t) - x_i^* \equiv 0, \qquad i = 1,\dots,n.$$

Hence, LaSalle's invariance principle ensures the global stability of x^*.

Now consider the interaction matrix A with the following form:

$$A = (a_{ij})_{n\times n} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \times & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \cdots & A_k \end{pmatrix}, \tag{9}$$

where each A_i $(i = 1,\dots,k)$ is a $r_i \times r_i$ matrix with $\sum_{i=1}^{k} r_i = n$ and irreducible, and all elements in the upper-right blocks of A are zero, and all matrices $\times$ in the left-lower have any elements.

Now consider the first subsystem composed of the first r_1 equations in (1),

$$\dot{y}_1 = \operatorname{diag}(y_1) A_1 (y_1 - y_1^*), \tag{10}$$

where $y_1 = (x_{11}, \ldots, x_{1r_1})$ and $y_1^* = (x_{11}^*, \ldots, x_{1r_1}^*)$. Since A_1 is irreducible, the above result implies that in M: $y_1(t) = y_1^*$ for $t \geq 0$. Substituting $y_1(t) = y_1^*$ into the second subsystem of (1) which is composed of r_2 equations from the $(r_1+1)^{\text{th}}$ one to the $(r_1+r_2)^{\text{th}}$ one in system (1), we have

$$\dot{y}_2 = \operatorname{diag}(y_2) A_2 (y_2 - y_2^*), \tag{11}$$

where $y_2 = (x_{21}, \ldots, x_{2r_2})$ and $y_2^* = (x_{21}^*, \ldots, x_{2r_2}^*)$. Since A_2 is irreducible, the above result again implies that in M: $y_2(t) = y_2^*$ for $t \geq 0$. Repeating the above procedure, we can obtain $M = \{x^*\}$.

This completes the proof of the theorem.

REFERENCES

1. B.S. Goh, *Management and Analysis of Biological Populations*, Elsevier Science, (1980).
2. J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, (1988).
3. Y. Takeuchi, *Global Dynamical Properties of Lotka-Volterra Systems*, World Scientific, Singapore, (1996).
4. J.P. LaSalle, Some extensions of Liapunov's second method, *IRE Trans. Circuit Theory* **7**, 520–527, (1960).